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A short note on the equivalence between 'best proximity' points and 'fixed point' results

Mohamed Jleli[1], Erdal Karapınar[2*] and Bessem Samet[1]

[*]Correspondence:
erdalkarapinar@yahoo.com;
erdal.karapinar@atilim.edu.tr
[2]Department of Mathematics, Atilim University, Ankara, 06836, Turkey
Full list of author information is available at the end of the article

Abstract
In this short note, we notice that, unexpectedly, some existing fixed point results and recently announced best proximity point results are equivalent.
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1 Introduction and preliminaries

In 1973 Geraghty [1] introduced the class S of functions $\beta : [0,\infty) \to [0,1)$ satisfying the following condition:

$$\beta(t_n) \to 1 \quad \text{implies} \quad t_n \to 0. \tag{1}$$

The author defined contraction mappings via functions from this class and proved the following result.

Theorem 1.1 (Geraghty [1]) *Let (X,d) be a complete metric space and $T : X \to X$ be an operator. If T satisfies the following inequality*:

$$d(Tx,Ty) \le \beta\big(d(x,y)\big)d(x,y) \quad \textit{for any } x,y \in X, \tag{2}$$

where $\beta \in S$, then T has a unique fixed point.

Theorem 1.1 was generalized in several ways, see *e.g.* [2–6]. Recently, Caballero *et al.* [2] introduced the following contraction.

Definition 1.1 ([7]) Let A, B be two nonempty subsets of a metric space (X,d). A mapping $T : A \to B$ is said to be a Geraghty-contraction if there exists $\beta \in S$ such that

$$d(Tx,Ty) \le \beta\big(d(x,y)\big)d(x,y) \quad \text{for any } x,y \in A. \tag{3}$$

For the sake of completeness, we recall some basic definitions and fundamental results.

Let (X,d) be a metric space and (A,B) a pair of nonempty subsets of X. We consider the following notations:

$$d(A,B) := \inf\{d(a,b) : a \in A, b \in B\};$$
$$d(x,B) := \inf\{d(x,b) : b \in B\};$$
$$A_0 := \{a \in A : d(a,b) = d(A,B) \text{ for some } b \in B\};$$
$$B_0 := \{b \in B : d(a,b) = d(A,B) \text{ for some } a \in A\}.$$

Through this paper, $\mathbb{N}$ denotes the set of natural numbers.

In [8], Sadiq Basha introduced the following concept.

Definition 1.2 We say that B is approximatively compact with respect to A if and only if every sequence $\{y_n\} \subset B$ satisfying the condition that $\lim_{n\to\infty} d(x,y_n) = d(x,B)$ for some x in A, has a convergent subsequence.

Definition 1.3 A mapping $g : A \to A$ is called an isometry if

$$d(gx,gy) = d(x,y), \quad \forall (x,y) \in A \times A.$$

Definition 1.4 (see *e.g.* [9]) Given a mapping $T : A \to B$ and an isometry $g : A \to A$, the mapping T is said to preserve the isometric distance with respect to g if and only if

$$d\big(T(gx), T(gy)\big) = d(Tx,Ty), \quad \forall (x,y) \in A \times A.$$

Denote by Ξ the set of functions $\varphi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(I) φ is continuous and nondecreasing;

(II) $\varphi(0) = 0$;

(III) $\lim_{t\to\infty} \varphi(t) = \infty$.

The following notions were introduced by Sadiq Basha [9].

Definition 1.5 A mapping $T : A \to B$ is said to be a generalized proximal contraction of the first kind if and only if

$$\left.\begin{aligned} d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{aligned}\right\} \quad \Longrightarrow \quad d(u,v) \le d(x,y) - \varphi\big(d(x,y)\big),$$

where $x,y,u,v \in A$ and $\varphi \in \Xi$.

Definition 1.6 A mapping $T : A \to B$ is said to be a generalized proximal contraction of the second kind if and only if

$$\left.\begin{aligned} d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{aligned}\right\} \quad \Longrightarrow \quad d(Tu,Tv) \le d(Tx,Ty) - \varphi\big(d(Tx,Ty)\big),$$

where $x,y,u,v \in A$ and $\varphi \in \Xi$.

Inspired by these definitions, Amini-Harandi [3] introduced the following definition.

Denote by Ψ the set of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(I) ψ is continuous and nondecreasing;
(II) $\psi(0) = 0$;
(III) $t \le \psi(t)$ for each $t \ge 0$.

Definition 1.7 A mapping $T : A \to B$ is said to be a generalized Geraghty proximal contraction of the first kind if and only if

$$\left.\begin{aligned} d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{aligned}\right\} \quad\Longrightarrow\quad \psi\bigl(d(u,v)\bigr) \le \beta\bigl(d(x,y)\bigr)\psi\bigl(d(x,y)\bigr),$$

where $x, y, u, v \in A$ and $\psi \in \Psi$, $\beta \in S$.

Definition 1.8 A mapping $T : A \to B$ is said to be a generalized Geraghty proximal contraction of the second kind if and only if

$$\left.\begin{aligned} d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{aligned}\right\} \quad\Longrightarrow\quad d(Tu,Tv) \le \beta\bigl(d(Tx,Ty)\bigr)\psi\bigl(d(Tx,Ty)\bigr),$$

where $x, y, u, v \in A$ and $\psi \in \Psi$, $\beta \in S$.

The main result in [3] is the following.

Theorem 1.2 *Let A and B be two nonempty closed subsets of a complete metric space (X,d) such that $A_0 \neq \emptyset$, $B_0 \neq \emptyset$ and B is approximatively compact with respect to A. Suppose that the mappings $g : A \to A$ and $T : A \to B$ satisfy the following conditions*:

(i) *T is a generalized Geraghty proximal contraction of the first kind*;
(ii) $T(A_0) \subseteq B_0$;
(iii) *g is an isometry*;
(iv) $A_0 \subseteq g(A_0)$.

Then there exists a unique element $x^* \in A$ such that

$$d\bigl(gx^*, Tx^*\bigr) = d(A,B).$$

Further, for any fixed element $x_0 \in A_0$, the iterative sequence $\{x_n\} \subset A_0$, defined by

$$d(gx_{n+1}, Tx_n) = d(A,B),$$

converges to x^*.

In this manuscript, we shall show that Theorem 1.2 is a particular case of existing fixed point theorems in the literature. Hence, the main result of [3] is not a real generalization.

2 Some useful results

Denote by Φ the set of functions $\phi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(I) ϕ is continuous and nondecreasing;

(II) $\phi(t) = 0$ if and only if $t = 0$.

First we show that we get the more general form of the main result in [3] by replacing the class of distance functions Ψ by Φ in Definition 1.7.

Theorem 2.1 *Let A and B be two nonempty closed subsets of a complete metric space (X,d) such that $A_0 \neq \emptyset$, $B_0 \neq \emptyset$, and B is approximatively compact with respect to A. Suppose that the mappings $g : A \to A$ and $T : A \to B$ satisfy the following conditions*:

(i)

$$\left.\begin{aligned} d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{aligned}\right\} \quad \Longrightarrow \quad \psi\bigl(d(u,v)\bigr) \le \beta\bigl(d(x,y)\bigr)\psi\bigl(d(x,y)\bigr),$$

where $x,y,u,v \in A$ and $\psi \in \Phi$, $\beta \in S$.

(ii) $T(A_0) \subseteq B_0$;

(iii) *g is an isometry*;

(iv) $A_0 \subseteq g(A_0)$.

Then there exists a unique element $x^* \in A$ such that

$$d\bigl(gx^*, Tx^*\bigr) = d(A,B).$$

Further, for any fixed element $x_0 \in A_0$, the iterative sequence $\{x_n\} \subset A_0$, defined by

$$d(gx_{n+1}, Tx_n) = d(A,B),$$

converges to x^*.

Proof By following the lines in the proof of Theorem 3.1 in the paper of Amini-Harandi [3], we conclude that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{4}$$

It is sufficient to prove that $\{x_n\}$ is a Cauchy sequence.

Suppose, on the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k) > m(k) > k$ and

$$d(x_{m(k)}, x_{n(k)}) \ge \varepsilon. \tag{5}$$

Furthermore, we can choose $n(k)$, associated with $m(k)$, is the smallest integer which satisfies $n(k) > m(k) > k$ and (5). Consequently, we have

$$d(x_{m(k)}, x_{n(k)-1}) < \varepsilon. \tag{6}$$

Due to Lemma 2.4, we conclude that

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \lim_{k\to\infty} d(x_{m(k-1)}, x_{n(k)-1}) = \varepsilon. \tag{7}$$

So, we obtain

$$\begin{aligned}\psi\bigl(d(x_{m(k)},x_{n(k)})\bigr) &= \psi\bigl(d(Tx_{m(k-1)},Tx_{n(k-1)})\bigr)\\ &\leq \beta\bigl(d(x_{m(k-1)},x_{n(k)-1})\bigr)\psi\bigl(d(x_{m(k-1)},x_{n(k)-1})\bigr)\\ &\leq \psi\bigl(d(x_{m(k-1)},x_{n(k)-1})\bigr).\end{aligned} \tag{8}$$

Letting $k \to \infty$ in the inequality above, we get

$$\psi(\varepsilon) \leq \beta\bigl(d(x_{m(k-1)},x_{n(k)-1})\bigr)\psi(\varepsilon) \leq \psi(\varepsilon),$$

since ψ is continuous and (7) holds. Due to the property of ψ, $\psi(\varepsilon) > 0$, we derive

$$\lim_{k\to\infty} \beta\bigl(d(x_{m(k-1)},x_{n(k)-1})\bigr) = 1 \tag{9}$$

from the last inequality above. Since $\beta \in S$, we conclude that

$$\lim_{k\to\infty} d(x_{m(k-1)},x_{n(k)-1}) = 0. \tag{10}$$

Due to (7) we get $\varepsilon = 0$, a contradiction. Hence, $\{x_n\}$ is Cauchy.

The rest follows from the corresponding lines in the proof of Theorem 3.1 in the paper of Amini-Harandi [3]. □

The following theorem is due to [10].

Theorem 2.2 *Let (X,d) be a complete metric space and $T : X \to X$ be an operator. If T satisfies the following inequality*:

$$\psi\bigl(d(Tx,Ty)\bigr) \leq \beta\bigl(d(x,y)\bigr)\psi\bigl(d(x,y)\bigr) \quad \textit{for any } x,y \in X, \tag{11}$$

where $\beta \in S$ and $\psi \in \Phi$, then T has a unique fixed point.

The following concept was introduced by Ćirić in [11].

Definition 2.1 Let (X,d) be a metric space and $f : X \to X$ be a self-mapping. We say that X is f-orbitally complete if and only if for any $x \in X$, if $\{f^n x\}$ is a Cauchy sequence, then it converges to some element in X.

It is evident that in Theorem 2.2, the notion of completeness of the metric space (X,d) can be replaced by the notion of f-orbitally completeness. Consequently, we derive the following fixed point result.

Lemma 2.1 (*cf.* [7]) *Let (X,d) be a f-orbitally complete metric space, where $f : X \to X$ is a self-mapping satisfying the following condition*:

$$\psi\bigl(d(fx,fy)\bigr) \leq \beta\bigl(d(x,y)\bigr)\psi\bigl(d(x,y)\bigr), \quad \forall(x,y) \in X \times X, \tag{12}$$

where $\psi \in \Psi$, $\beta \in S$. Then f has a unique fixed point $x^* \in X$. Moreover, for any $x \in X$, the sequence $\{f^n x\}$ converges to x^*.

Regarding the analogy with the proof of Lemma 2.2 in [7].

Lemma 2.2 (*cf.* [7]) *Let (A,B) be a pair of closed subsets of a metric space (X,d). Suppose that the following conditions hold*:

(i) $A_0 \neq \emptyset$;
(ii) *B is approximatively compact with respect to A.*

Then the set A_0 is closed.

Lemma 2.3 (*cf.* [7]) *Let A and B be nonempty subsets of a metric space (X,d) such that $A_0 \neq \emptyset$. Suppose that the mappings $g : A \to A$ and $T : A \to B$ satisfy the following conditions*:

(i) *T is a generalized Geraghty-proximal contraction of the first kind*;
(ii) $T(A_0) \subseteq B_0$;
(iii) *g is an isometry*;
(iv) $A_0 \subseteq g(A_0)$.

Then there exists a self-mapping $f : A_0 \to A_0$ satisfying the condition:

$$d\big(Tx, g(fx)\big) = d(A,B), \quad \forall x \in A_0. \tag{13}$$

Regarding the analogy with the proof of Lemma 2.4 in [7].

Lemma 2.4 ([12]) *Let (X,d) be a metric space and let (x_n) be a sequence in X such that $(d(x_{n+1},x_n))$ is non-increasing and*

$$\lim_{n\to+\infty} d(x_{n+1},x_n) = 0.$$

If (x_n) is not a Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences (m_k) and (n_k) of positive integers such that $m_k > n_k \geq k$ and the following four sequences tend to ε when $k \to +\infty$:

$$\big(d(x_{m_k},x_{n_k})\big), \qquad \big(d(x_{n_k},x_{m_k+1})\big), \qquad \big(d(x_{m_k},x_{n_k+1})\big), \qquad \big(d(x_{m_k+1},x_{n_k+1})\big).$$

3 Main results

Theorem 3.1 *Theorem* 2.1 *is a consequence of Theorem* 2.2.

Proof Suppose that all the assumptions of Theorem 1.2 are satisfied. From Lemma 2.3, there exists a self-mapping $f : A_0 \to A_0$ satisfying (13). Then, for all $(x,y) \in A_0 \times A_0$, we have

$$\begin{cases} d(g(fx), Tx) = d(A,B), \\ d(g(fy), Ty) = d(A,B). \end{cases}$$

Since T is a generalized Geraghty-proximal contraction of the first kind and g is an isometry, we obtain

$$\psi\big(d(fx,fy)\big) = \psi\big(d\big(g(fx),g(fy)\big)\big) \leq \beta\big(d(x,y)\big)\psi\big(d(x,y)\big),$$

for every pair $(x,y) \in A_0 \times A_0$. Thus f satisfies inequality (11).

Since (X,d) is complete, (A_0,d) is also complete. From Theorem 2.2, the self-mapping $h: A_0 \to A_0$ has a unique fixed point $x^* \in A_0$.

Note that from (13), since T is a generalized Geraghty-proximal contraction of the first kind and g is an isometry, we have $x^* \in A_0$ is a fixed point of f if and only if $x^* \in A$ and $d(gx^*, Tx^*) = d(A,B)$. Then there exists a unique $x^* \in A_0$ such that $d(gx^*, Tx^*) = d(A,B)$. Now, let $a \in A_0$ be an arbitrary point. Consider a sequence $\{a_n\} \subset A_0$ satisfying

$$d(ga_{n+1}, Ta_n), \quad \forall n \in \mathbb{N}.$$

Since T is a generalized Geraghty-proximal contraction of the first kind and g is an isometry, it follows from (13) that

$$a_{n+1} = fa_n, \quad \forall n \in \mathbb{N}.$$

From Lemma 2.1, we have

$$\lim_{n\to\infty} d\big(a_n, x^*\big) = 0.$$

This ends the proof. □

4 Consequences

Theorem 4.1 *Let A and B be two nonempty closed subsets of a complete metric space (X,d) such that $A_0 \neq \emptyset$ and $B_0 \neq \emptyset$. Suppose that the mappings $g: A \to A$ and $T: A \to B$ satisfy the following conditions*:

(i) *T is a generalized Geraghty proximal contraction of the first and second kind,*

$$\left.\begin{aligned} d(u,Tx) = d(A,B)\\ d(v,Ty) = d(A,B)\end{aligned}\right\} \quad \Longrightarrow \quad \psi\big(d(u,v)\big) \le \beta\big(d(x,y)\big)\psi\big(d(x,y)\big),$$

$$\left.\begin{aligned} d(u,Tx) = d(A,B)\\ d(v,Ty) = d(A,B)\end{aligned}\right\} \quad \Longrightarrow \quad d(Tu,Tv) \le \beta\big(d(Tx,Ty)\big)\psi\big(d(Tx,Ty)\big),$$

where $x,y,u,v \in A$ and $\psi \in \Phi$, $\beta \in S$;

(ii) $T(A_0) \subseteq B_0$;

(iii) *g is an isometry*;

(iv) $A_0 \subseteq g(A_0)$.

Then there exists a unique element $x^* \in A$ such that

$$d\big(gx^*, Tx^*\big) = d(A,B).$$

Further, for any fixed element $x_0 \in A_0$, the iterative sequence $\{x_n\} \subset A_0$, defined by

$$d(gx_{n+1}, Tx_n) = d(A,B),$$

converges to x^*.

Theorem 4.2 *Theorem* 4.1 *is a consequence of Lemma* 2.1.

Proof Suppose that all the assumptions of Theorem 4.1 are satisfied. From Lemma 2.3, there exists a self-mapping $f : A_0 \to A_0$ satisfying (13). Then, for all $(x,y) \in A_0 \times A_0$, we have

$$\begin{cases} d(g(fx), Tx) = d(A,B), \\ d(g(fy), Ty) = d(A,B). \end{cases}$$

Since T is a generalized Geraghty-proximal contraction of the first kind and g is an isometry, we obtain

$$\psi\big(d(fx,fy)\big) = \psi\big(d\big(g(fx),g(fy)\big)\big) \le \beta\big(d(x,y)\big)\psi\big(d(x,y)\big),$$

for every pair $(x,y) \in A_0 \times A_0$. Thus f satisfies inequality (11).

Now, we shall prove that (A_0,d) is f-orbitally complete. Indeed, let $x_0 \in A_0$ and consider the sequence $\{x_n\} \subset A_0$ defined by $x_n = f^n x_0$ for all $n \in \mathbb{N}$. Suppose that $\{x_n\}$ is a Cauchy sequence, we have to prove that $\{x_n\}$ converges to some element in A_0. Since (X,d) is complete and A is closed, there exists some $z \in A$ such that

$$\lim_{n\to\infty} d(x_n,z) = 0. \tag{14}$$

By the definition of f, for all $n \in \mathbb{N}$, we have

$$\begin{cases} d(g(fx_n), Tx_n) = d(A,B), \\ d(g(fx_{n+1}), Tx_{n+1}) = d(A,B), \end{cases}$$

which implies (since T is a generalized Geraghty-proximal contraction of the second kind) that

$$\begin{aligned} \psi\big(d\big(T(gx_{n+1}), T(gx_{n+2})\big)\big) &= \psi\big(d\big(T\big(g(fx_n)\big), T\big(g(fx_{n+1})\big)\big)\big) \\ &\le \beta\big(d(Tx_n, Tx_{n+1})\big)\psi\big(d(Tx_n, Tx_{n+1})\big), \end{aligned}$$

for all $n \in \mathbb{N}$. Since T preserves the isometric distance with respect to g, we obtain

$$\psi\big(d(Tx_{n+1}, Tx_{n+2})\big) \le \beta\big(d(Tx_n, Tx_{n+1})\big)\psi\big(d(Tx_n, Tx_{n+1})\big), \quad \forall n \in \mathbb{N}.$$

Following the same lines as the proof of Theorem 3.1 in [3], one can show that $\{Tx_n\}$ is a Cauchy sequence in the complete metric space (X,d). Since B is closed, there exists some $b \in B$ such that

$$\lim_{n\to\infty} d(Tx_n, b) = 0. \tag{15}$$

Now, using (14), (15), and the definition of f, we get

$$d(A,B) = \lim_{n\to\infty} d\big(g(fx_n), Tx_n\big) = \lim_{n\to\infty} d(gx_{n+1}, Tx_n) = d(gz, b).$$

Note that since g is an isometry, it is continuous. Now, we have

$$d(A,B) = d(gz,b), \quad b \in B.$$

This implies that $gz \in A_0$. On the other hand, since $A_0 \subseteq g(A_0)$ and g is an isometry, we obtain $z \in A_0$. Thus, we proved that A_0 is f-orbitally complete. Now, applying Lemma 2.1, we find that f has a unique fixed point $x^* \in A_0$.

The rest follows from the lines of the proof of Theorem 3.1. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, King Saud University, Riyadh, Saudi Arabia. [2]Department of Mathematics, Atilim University, Ankara, 06836, Turkey.

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